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NOTE

A Divergence Theorem for a Nonlinear Dirichlet Problem[1]

Consider the Dirichlet problem

$$\begin{aligned} \Delta u &= f(u), \quad \text{in} \quad G, \\ u &= \phi, \quad \text{on} \quad \Gamma, \end{aligned} \tag{1}$$

where Δ is the Laplacian, f may be nonlinear, ϕ is a continuous function and G is a bounded domain in R^n with piecewise smooth boundary Γ. Under appropriate conditions on f, it is known [1]–[3] that this problem and certain of its finite difference analogues have unique solutions. The purpose of this note is to establish, for a finite difference analogue of (1), a condition on the mesh size under which a particular successive approximation scheme fails to converge to the solution.

If we discretize G into a lattice with equal spacing h, one finite difference analogue of (1) can be written as

$$\begin{aligned} \Delta_h[u_i] &= f(u_i), \qquad i = 1,\dots, N, \\ u_j &= \phi_j, \qquad j = N+1,\dots, M, \end{aligned} \tag{2}$$

where Δ_h is the central difference operator corresponding to Δ and u_i and u_j are the values of u at interior and boundary nodal points.

One method of solving (2) is by taking an initial guess, $u^{(0)}$, for u and defining a sequence of approximations, $\{u^{(k)}\}$, by

$$\begin{aligned} \Delta_h[u_i^{(k)}] &= f(u_i^{(k-1)}), \qquad i = 1,\dots, N, \\ u_j &= \phi_j, \qquad j = N+1,\dots, M; \qquad k \geqslant 1. \end{aligned} \tag{3}$$

THEOREM. *If*

$$f'(u) \geqslant \gamma,$$

for some positive constant γ, *then* $\{u^{(k)}\}$ *does not converge to the solution of* (2) *when*

$$h^2 > \frac{4n}{\gamma},$$

where n *is the dimension of* R^n.

[1] This work was performed under the auspices of the U.S. Atomic Energy Commission while the author was at the Applied Mathematics Department, Brookhaven National Laboratory, Upton, New York.

Proof. We shall prove the result for $n = 1$; the cases for $n \geqslant 2$ follow in an analogous manner.

Denote the solution of (2) by $u = (u_1, \ldots, u_N)$. Suppose $\epsilon^{(k)} = u - u^{(k)}$. Subtracting the matrix form of (3) from the matrix form of (2) gives

$$M_k^{-1}\epsilon^{(k)} = \epsilon^{(k-1)},$$

where

$$M_k^{-1} = \begin{bmatrix} 2\alpha_1^{-1} & -\alpha_1^{-1} & & & 0 \\ -\alpha_2^{-1} & 2\alpha_2^{-1} & -\alpha_2^{-1} & & \\ & & \cdots & & \\ & -\alpha_{N-1}^{-1} & 2\alpha_{N-1}^{-1} & -\alpha_{N-1}^{-1} & \\ 0 & & -\alpha_N^{-1} & 2\alpha^{-1}{}_N \end{bmatrix},$$

$$\alpha_i = -h^2 f'(\xi_i^{(k)}),$$

and $\xi_i^{(k)}$ is between $u_i^{(k)}$ and u_i.

Since M_k^{-1} is irreducibly diagonally dominant, it is nonsingular (see [4]). Therefore, M_k exists and

$$\rho(M_k) \geqslant \frac{1}{\rho(M_k^{-1})} \geqslant \frac{1}{4 \max_{1 \leqslant i \leqslant N} |\alpha_i^{-1}|} \geqslant \frac{\gamma h^2}{4},$$

where $\rho(M_k)$ denotes the spectral radius of M_k. For $h^2 > 4/\gamma$, $\rho(M_k) > 1$, which implies divergence.

Specific numerical applications of this theorem are discussed in [5].

References

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